Int. J. Solids Structures Vol. 30, No. 1, pp. 19–36, 1993
Printed in Great Britain.

0020–7683/93 $5.00+ .00

MICROMECHANICAL MODELING OF DAMAGE IN UNIAXIALLY LOADED UNIDIRECTIONAL FIBER-REINFORCED COMPOSITE LAMINAE

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(*Received* 13 *May* 1991; *in revised form* 5 *June* 1992)

Abstract—A micromechanical composite model is used to study damage in a uniaxially loaded unidirectional fiber-reinforced composite thin lamina. The matrix and fiber materials are assumed to be elastic with the fibers aligned. An overall damage variable is introduced based on the concept of effective stress. The local damage effects are modeled through two additional separate damage variables which represent matrix and fiber damage. A local–overall relation for the damage variables is derived.

Stress and strain concentration factors are derived for the damaged composite. Damage evolution is also considered using both local and overall analyses based on an extremum principle.

1. INTRODUCTION

Composite materials play an increasingly important role in the industry today. Of particular importance is the problem of initiation and evolution of damage in fiber-reinforced composites. The analysis of damage mechanisms in two-phase composites is a rather complex problem that has challenged researchers during the past two decades. In particular, the literature lacks a consistent and systematic approach to the study of damage in composite materials.

In reviewing the available literature concerning fiber-reinforced composites, it is clear that two different approaches are employed. In the first approach, the composite material is treated as a transversely isotropic medium and continuum theories are used in its analysis [see for example, Talreja (1985, 1986), Christensen (1988, 1990), Shen *et al.* (1985) and Lene (1986)]. In this approach, the fiber direction is taken as the direction of anisotropy and the classical equations of anisotropic elasticity are used throughout. The disadvantages of this approach are that no distinction is made regarding the different phases in the analysis of stresses and strains and no consideration is given to the local effects of deformation and damage. Other researchers (Badaliance *et al.*, 1977) used fracture mechanics techniques to analyse cracks in multilayered plates.

In the second approach, micromechanical models are used where the matrix and fibers are treated separately in a local analysis and this, in turn, is linked with the overall composite behavior. Different micromechanical models employ different methods of achieving the local–overall relations. Hill (1965, 1972) employed volume averages of stress and strain increments in the different phases and introduced certain concentration factors to relate these volume averages of the local fields to the overall uniform increments. Dvorak and Bahei-El-Din (1979, 1982, 1987) and Bahei-El-Din and Dvorak (1989) used Hill's technique to analyse the elasto-plastic behavior of fiber-reinforced composites where they considered elastic fibers and an elasto-plastic matrix. In their micromechanical analysis of elasto-plastic composites, Dvorak and Bahei-El-Din (1987) identified two distinct deformation modes. One is matrix dominated and the other is fiber dominated. The first mode is prevalent in the case of stiff elastic fibers, while the second mode is more general where the elastic fibers are more compliant and the mode is treated as a general case of plastic deformation of a heterogeneous medium. Aboudi (1990) used an averaging technique in order to relate the local stresses to the overall composite stress.

A thermomechanical constitutive theory has recently been proposed by Allen and Harris (1987) and Allen *et al.* (1987) to analyse distributed damage in elastic composites. In particular, the problem of matrix cracking has been extensively studied in the literature

(Dvorak *et al.*, 1985; Dvorak and Laws, 1987; Laws and Dvorak, 1987; Allen *et al.*, 1988; Lee *et al.*, 1989).

Continuum damage mechanics appeared for the first time in 1958 when Kachanov (1958) introduced the concept of effective stress. Research in this area progressed rapidly based on Kachanov's work and has now reached a stage where practical engineering applications are possible. Lemaitre (1985, 1986), Chaboche (1988a, b) and Krajcinovic (1983, 1984) used continuum damage mechanics to analyse different types of damage ranging from brittle fracture to ductile failure. However, the application of continuum damage mechanics to composite materials has been restricted to models of composites utilizing a transversely isotropic medium (Talreja, 1985). It is clear that such an approach is not sophisticated enough to account for local effects and no distinction is made between matrix damage and fiber damage, or even for damage resulting from the matrix–fiber interaction.

In this work, a micromechanical composite model is used to study damage in a uniaxially loaded unidirectional fiber-reinforced composite thin lamina. This research constitutes a first step toward development of a consistent, micromechanically based damage theory for composite materials. A two-phase elastic composite (matrix and fibers) is considered. Local damage variables are introduced within the framework of the effective stress concept (Kachanov, 1958). The local damage variables are then related to an overall damage variable. Stress and strain concentration factors for the damaged material are derived in terms of the undamaged concentration factors (Dvorak and Bahei-El-Din, 1979; Bahei-El-Din and Dvorak, 1989) and the damage variables. Finally, a criterion for damage evolution is proposed based on the works of Lemaitre (1985), Lee *et al.* (1985) and Kattan and Voyiadjis (1990). The local–overall relation for damage evolution is derived based on micromechanical considerations. An extremum principle is used to formulate the criterion for damage evolution. The uniaxial loading problem is investigated here in order to explore the physical interpretation of the proposed theory as far as possible. A generalization of this theory is possible, however the tools of tensor analysis are needed and will be discussed in a subsequent paper.

2. CHARACTERIZATION OF DAMAGE: LOCAL VS OVERALL DAMAGE

Kachanov (1958) introduced the idea of effective stress in order to characterize damage initiation and evolution within the framework of the mechanics of continuous media. In this approach, a damage variable is defined and used to represent degradation of the material which reflects various types of damage at the micro-scale level like nucleation and growth of voids, cavities, micro-cracks and other microscopic defects.

In the case of composite materials, the damage variable will also reflect the additional types of damage that occur in these materials like fracture of fibers, debonding and delamination, etc. In the following, an overall damage variable is introduced for the whole composite system. This damage variable is found to be decomposable into two local damage variables that are directly related to the matrix and fibers.

In this work, the discussion is limited to damage due to uniaxial tension in a unidirectional fiber-reinforced composite thin lamina. This is done deliberately in order to keep the mathematical formulation simple and accessible to the general reader. Analysis of general states of damage and deformation in composite materials will require the use of tensor analysis and will be left to a subsequent paper.

2.1. *Stresses*

Consider a unidirectional fiber-reinforced composite thin lamina that is subjected to a uniaxial tensile force T along the x_1-direction as shown in Fig. 1(a). Both the matrix and fibers are assumed to be linearly elastic with the fibers being continuous, aligned and symmetrically distributed along the x_1-axis. Let dA be the cross-sectional area of the lamina with dA^M and dA^F being the cross-sectional areas of the matrix and fibers, respectively (superscripts "M" and "F" are used throughout the manuscript to denote matrix- and fiber-related quantities, respectively). Since the composite lamina is assumed to consist of

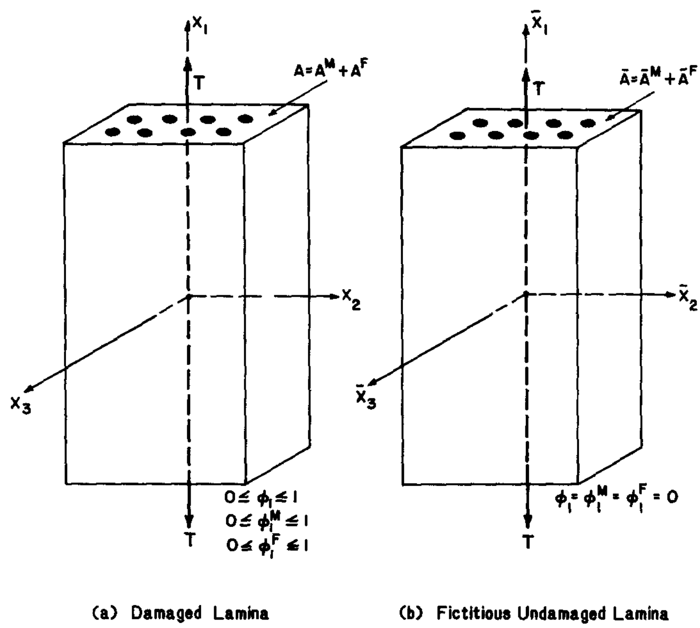

Fig. 1. Damage due to uniaxial tension.

two phases only, it is clear that $dA = dA^M + dA^F$. The overall stress increment $d\sigma$ is clearly T/dA and the local stress increments $d\sigma^M$ and $d\sigma^F$ are related to the overall stress increment $d\sigma$ by

$$d\sigma = c^M \, d\sigma^M + c^F \, d\sigma^F, \tag{1}$$

where c^M and c^F are the matrix and fiber volume fractions (or area functions here) given by dA^M/dA and dA^F/dA, respectively. It should be clear to the reader that $c^M + c^F = 1$. The local transverse stress increments $d\sigma_2^M$, $d\sigma_2^F$, $d\sigma_3^M$ and $d\sigma_3^F$, although nonzero, are not considered in this work.

Using the concept of effective stress, one now considers a fictitious lamina [see Fig. 1(b)] made of the same composite material described above and subjected to the same uniaxial tensile force T. This lamina is assumed to undergo deformation with no damage. In other words, it can be hypothetically obtained from the lamina in Fig. 1(a) by removing all the damage that the lamina has experienced. Let $d\bar{A}$ denote the cross-sectional area of the undamaged lamina with $d\bar{A}^M$ and $d\bar{A}^F$ denoting the cross-sectional areas of the undamaged matrix and fibers, respectively. These quantities represent net or effective areas that include no damage. Also, let $\bar{c}^M$ and $\bar{c}^F$ denote the volume (or area) fractions for the undamaged matrix and fibers, respectively. The following relations should be clear:

$$d\bar{A}^M + d\bar{A}^F = d\bar{A}, \quad d\bar{A} \leqslant dA, \quad d\bar{A}^M \leqslant dA^M \quad \text{and} \quad d\bar{A}^F \leqslant dA^F.$$

The overall effective stress increment $d\bar{\sigma}$ is taken to be the stress in the fictitious lamina and it is clear that $d\bar{\sigma} = T/d\bar{A}$. One also considers the two local effective stress increments $d\bar{\sigma}^M$ and $d\bar{\sigma}^F$ and as before, it can be shown that they are related to $d\bar{\sigma}$ by

$$d\bar{\sigma} = \bar{c}^F \, d\bar{\sigma}^F + \bar{c}^M \, d\bar{\sigma}^M. \tag{2}$$

Since the two laminae are assumed to be mechanically equivalent (in terms of the uniaxial force T that is applied to each one), it follows directly that $d\bar{\sigma} = d\sigma \, dA/d\bar{A}$. The ratio of the damaged area $dA - d\bar{A}$ to the original area dA is now used to define an overall damage variable ϕ_1 in the x_1-direction as follows:

$$\phi_1 = \frac{\mathrm{d}A - \mathrm{d}\bar{A}}{\mathrm{d}A}. \tag{3}$$

It is clear that the values of ϕ_1 range from 0 for undamaged material to 1 for (theoretically) complete rupture. The effective stress increment can now be written in terms of the damage variable

$$\mathrm{d}\bar{\sigma} = \frac{\mathrm{d}\sigma}{1-\phi_1}. \tag{4}$$

The above expression has been used extensively in the literature (Kachanov, 1958; Lemaitre, 1985, 1986; Chaboche, 1988a, b) to model various types of phenomena like ductile failure, brittle fracture, creep, etc.

In order to represent local damage effects in the marix and fibers, one defines two additional (local) damage variables ϕ_1^{M} and ϕ_1^{F}. The first one ϕ_1^{M} is used to model damage in the matrix like nucleation, growth and coalescence of voids and microcracks, etc., while the second one ϕ_1^{F} is used to model damage in the fiber and that due to fiber–matrix interaction such as fiber fracture, debonding, etc. These two variables are defined as before based on the ratios of the relevant cross-sectional areas of the matrix and fibers as follows:

$$\phi_1^{\mathrm{M}} = \frac{\mathrm{d}A^{\mathrm{M}} - \mathrm{d}\bar{A}^{\mathrm{M}}}{\mathrm{d}A^{\mathrm{M}}}, \tag{5a}$$

$$\phi_1^{\mathrm{F}} = \frac{\mathrm{d}A^{\mathrm{F}} - \mathrm{d}\bar{A}^{\mathrm{F}}}{\mathrm{d}A^{\mathrm{F}}}. \tag{5b}$$

It is clear from eqns (5) that the local damage variables satisfy the inequalities $0 \leqslant \phi_1^{\mathrm{M}} \leqslant 1$ and $0 \leqslant \sigma_1^{\mathrm{F}} \leqslant 1$.

One can now derive equations for the effective matrix and fiber volume fractions $\bar{c}^{\mathrm{M}}$ and $\bar{c}^{\mathrm{F}}$ in terms of c^{M} and c^{F} and the damage variables ϕ_1^{M} and ϕ_1^{F}. Starting with $\bar{c}^{\mathrm{M}} = \mathrm{d}\bar{A}^{\mathrm{M}}/\mathrm{d}\bar{A}$ and $\bar{c}^{\mathrm{F}} = \mathrm{d}\bar{A}^{\mathrm{F}}/\mathrm{d}\bar{A}$ along with eqns (3) and (5), one can show that

$$\bar{c}^{\mathrm{M}} = c^{\mathrm{M}}\frac{1-\phi_1^{\mathrm{M}}}{1-\phi_1}, \tag{6a}$$

$$\bar{c}^{\mathrm{F}} = c^{\mathrm{F}}\frac{1-\phi_1^{\mathrm{F}}}{1-\phi_1}. \tag{6b}$$

Also, using eqn (3) along with eqns (5), one derives

$$\phi_1\,\mathrm{d}A = \phi_1^{\mathrm{M}}\,\mathrm{d}A^{\mathrm{M}} + \phi_1^{\mathrm{F}}\,\mathrm{d}A^{\mathrm{F}}. \tag{7}$$

Dividing eqn (7) through by $\mathrm{d}A$, one derives the relationship between the local damage variables ϕ_1^{M} and ϕ_1^{F} and the overall damage variable ϕ_1 as follows:

$$\phi_1 = c^{\mathrm{M}}\phi_1^{\mathrm{M}} + c^{\mathrm{F}}\phi_1^{\mathrm{F}}. \tag{8}$$

The relationship between the matrix damage ratio $\phi_1^{\mathrm{M}}/\phi_1$ and the fiber damage ratio $\phi_1^{\mathrm{F}}/\phi_1$ is shown in Fig. 2 for different values of the matrix volume fraction c^{M}. It is clear from the figure that these ratios are always greater than or equal to one, implying that $\phi_1^{\mathrm{M}} \geqslant \phi_1$ and $\phi_1^{\mathrm{F}} \geqslant \phi_1$. This remark does not contradict the fact that the matrix and fiber damage should be a part of the composite damage since the damage variables are defined as ratios of areas and do not reflect the absolute amount of damage in the material.

Adding eqns (6a) and (6b) and utilizing eqn (8) and the previous relation between c^M and c^F, one concludes that $\bar{c}^M + \bar{c}^F = 1$, that is, the phase volume fractions of the damaged material satisfy the same relation as that of the undamaged material indicating no significant (or large) changes in the geometry of the composite system. Some authors use the "continuity" variable ψ_1 defined by $\psi_1 = 1 - \phi_1$ (e.g. Kachanov, 1986). In this case, one can easily show that ψ_1 satisfies a relation similar to that of eqn (8), namely, $\psi_1 = c^M \psi_1^M + c^F \psi_1^F$ where $\psi_1^M = 1 - \phi_1^M$ and $\psi_1^F = 1 - \phi_1^F$.

Substituting eqns (4) and (6) into eqn (2) and simplifying, one obtains the following relation for the effective local stress increments $d\bar{\sigma}^M$ and $d\bar{\sigma}^F$:

$$d\sigma = c^M(1 - \phi_1^M)\, d\bar{\sigma}^M + c^F(1 - \phi_1^F)\, d\bar{\sigma}^F. \tag{9}$$

In the derivation of eqn (9), it is assumed that $\phi_1 \neq 1$. Therefore, the case of complete rupture is excluded from the discussion that follows. In view of the effective stress equation (4), one can assume similar expressions for the effective local stresses as follows:

$$d\bar{\sigma}^M = \frac{d\sigma^M}{1 - \phi_1^M}, \tag{10a}$$

$$d\bar{\sigma}^F = \frac{d\sigma^F}{1 - \phi_1^F}. \tag{10b}$$

It is clear that eqns (10) satisfy the requirement given by eqn (9). However, the constraint given in eqn (9) is a necessary condition to be satisfied by any alternative expression for the effective local stresses other than eqns (10). Next, one considers the relations between the local and overall stresses in the composite system. Following the work of Dvorak and Bahei-El-Din (1979, 1982) and Bahei-El-Din and Dvorak (1989), one considers a micromechanically based approach and introduces the matrix and fiber stress concentration factors B^M and B^F in the undamaged lamina as given in Fig. 1(b). Therefore, one can write the following local–overall relations for the effective stress increments:

$$d\bar{\sigma}^M = B^M\, d\bar{\sigma}, \tag{11a}$$

$$d\bar{\sigma}^F = B^F\, d\bar{\sigma}. \tag{11b}$$

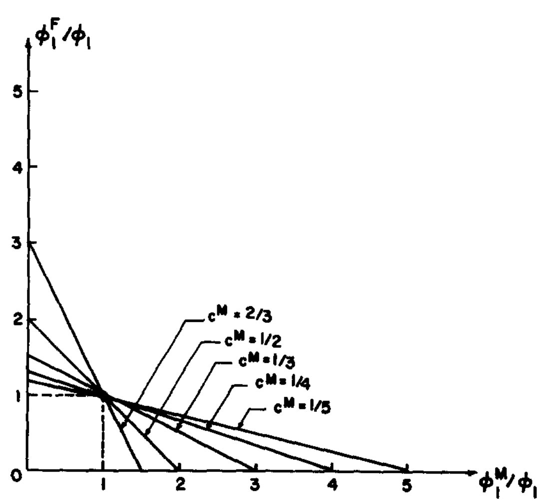

Fig. 2. Relationship between local damage parameters ϕ_1^M/ϕ_1 and ϕ_1^F/ϕ_1 for different matrix volume fractions.

The stress concentration factors B^{M} and B^{F} can be derived from the solution of an inclusion problem in the undamaged material. However, certain models have been proposed by Dvorak and Bahei-El-Din (1979, 1982) in order to derive simple expressions for B^{M} and B^{F}. Two of these models will be discussed at the end of Section 2.3 as they relate to the problem at hand.

Substituting eqns (11) into eqn (2), one obtains the relation between the stress concentration factors and the effective volume fractions

$$\bar{c}^{M}B^{M}+\bar{c}^{F}B^{F} = 1. \tag{12}$$

Substituting further for $\bar{c}^{M}$ and $\bar{c}^{F}$ from eqns (6) into eqn (12), one obtains:

$$c^{M}(1-\phi_{1}^{M})B^{M}+c^{F}(1-\phi_{1}^{F})B^{F} = 1-\phi_{1}. \tag{13}$$

Assuming that stress concentration factors $\bar{B}^{M}$ and $\bar{B}^{F}$ exist in the actual damaged lamina, one can write the following local–overall relations for the corresponding stress increments:

$$\mathrm{d}\sigma^{M} = \bar{B}^{M}\,\mathrm{d}\sigma, \tag{14a}$$

$$\mathrm{d}\sigma^{F} = \bar{B}^{F}\,\mathrm{d}\sigma. \tag{14b}$$

Substituting eqns (14) into eqn (1), one obtains the relation between the volume fractions and the damaged stress concentration factors [see eqn (12) for comparison];

$$c^{M}\bar{B}^{M}+c^{F}\bar{B}^{F} = 1. \tag{15}$$

Finally, one substitutes eqns (11) into eqns (10) along with eqn (4). Comparing the resulting two equations with eqns (14), one concludes that the damaged stress concentration factors are given by

$$\bar{B}^{M} = B^{M}\frac{1-\phi_{1}^{M}}{1-\phi_{1}}, \tag{16a}$$

$$\bar{B}^{F} = B^{F}\frac{1-\phi_{1}^{F}}{1-\phi_{1}}. \tag{16b}$$

Therefore, once appropriate expressions are derived for the undamaged stress concentration factors B^{M} and B^{F}, one can use eqns (16) to derive the corresponding expressions for the damaged stress concentration factors $\bar{B}^{M}$ and $\bar{B}^{F}$.

The relation given in eqn (16a) is now investigated in Figs 3 and 4. In Fig. 3, the relation between the matrix damage variable ϕ_{1}^{M} and the ratio $\bar{B}^{M}/B^{M}$ is shown for different values of the overall damage variable ϕ_{1}. It is noticed that the damaged matrix stress concentration factor becomes larger (i.e. the ratio $\bar{B}^{M}/B^{M}$ grows) with the decrease in the matrix damage variable ϕ_{1}^{M}. This is also clear in Fig. 4. However, Fig. 4 also shows that $\bar{B}^{M}/B^{M}$ increases with the increase in the overall damage variable ϕ_{1}. Similar remarks apply for the fiber stress concentration ratio $\bar{B}^{F}/B^{F}$ of eqn (16b).

2.2. *Strains*

In this section, the appropriate expressions for the effective strain increments $\mathrm{d}\bar{\varepsilon}_{1}$, $\mathrm{d}\bar{\varepsilon}_{2}$ and $\mathrm{d}\bar{\varepsilon}_{3}$ will be developed in terms of the strain increments $\mathrm{d}\varepsilon_{1}$, $\mathrm{d}\varepsilon_{2}$ and $\mathrm{d}\varepsilon_{3}$, and the damage variables ϕ_{1}, ϕ_{2} and ϕ_{3} (ϕ_{2} and ϕ_{3} are overall transverse damage variables along the x_{2}- and x_{3}-directions, respectively). In addition, the local–overall strain equations will be derived for both the damaged and undamaged materials.

In order to derive the required relations, the hypothesis of elastic energy equivalence (Sidoroff, 1981) is used. In this hypothesis, it is assumed that the elastic energy for a damaged material is equivalent in form to that of the undamaged material except that the

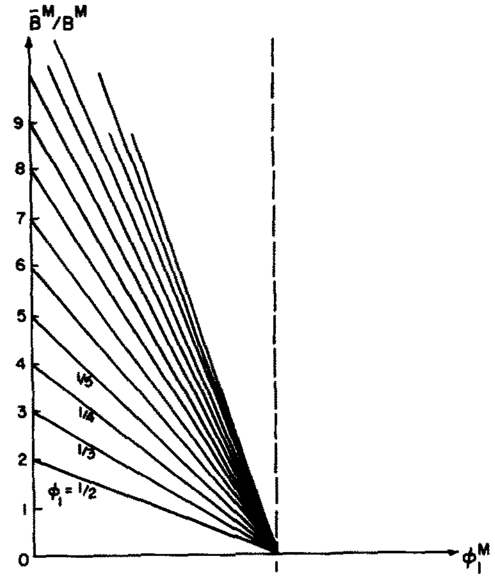

Fig. 3. Effect of matrix damage ϕ_1^M on the stress concentration factor for different overall damage parameters ϕ_1.

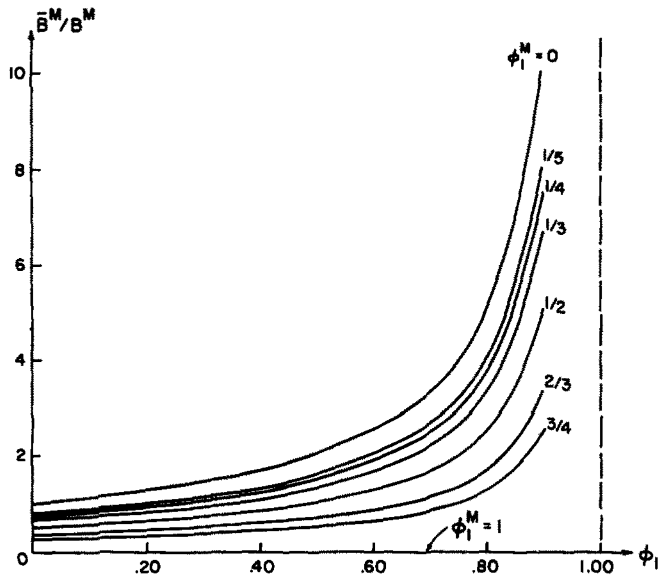

Fig. 4. Effect of overall damage ϕ_1 on the stress concentration factor for different matrix damage parameters ϕ_1^M.

stress is replaced by the effective stress in the energy formulation. Applying this to the overall composite system considered here, this hypothesis takes the following form:

$$\tfrac{1}{2}\,\mathrm{d}\sigma\,\mathrm{d}\varepsilon_1 = \tfrac{1}{2}\,\mathrm{d}\bar{\sigma}\,\mathrm{d}\bar{\varepsilon}_1, \tag{17}$$

where $\mathrm{d}\varepsilon_1$ is the overall axial strain increment in the x_1-direction and $\mathrm{d}\bar{\varepsilon}_1$ is its effective counterpart.

Substituting for $\mathrm{d}\bar{\sigma}$ from eqn (4) into eqn (17), one obtains the following expression for the effective overall axial strain increment $\mathrm{d}\bar{\varepsilon}_1$:

$$d\bar{\varepsilon}_1 = (1-\phi_1)\, d\varepsilon_1. \tag{18}$$

In view of the above relation, one can assume similar relations for the transverse overall strain increments $d\varepsilon_2$ and $d\varepsilon_3$:

$$d\bar{\varepsilon}_2 = (1-\phi_2)\, d\varepsilon_2, \tag{19a}$$

$$d\bar{\varepsilon}_3 = (1-\phi_3)\, d\varepsilon_3, \tag{19b}$$

where ϕ_2 and ϕ_3 are the overall transverse damage variables. The reader should note that definitions for ϕ_2 and ϕ_3 similar to the definition of ϕ_1 in eqn (3) are not possible. A more suitable way to define these two variables is suggested in the next section.

Next, the local–overall strain relations are discussed. The matrix and fiber axial strain increments are related to the overall axial strain increment in the fictitious undamaged state by the following relations:

$$d\bar{\varepsilon}_1^{\mathrm{M}} = C_{11}^{\mathrm{M}}\, d\bar{\varepsilon}_1 + C_{12}^{\mathrm{M}}\, d\bar{\varepsilon}_2 + C_{13}^{\mathrm{M}}\, d\bar{\varepsilon}_3, \tag{20a}$$

$$d\bar{\varepsilon}_1^{\mathrm{F}} = C_{11}^{\mathrm{F}}\, d\bar{\varepsilon}_1 + C_{12}^{\mathrm{F}}\, d\bar{\varepsilon}_2 + C_{13}^{\mathrm{F}}\, d\bar{\varepsilon}_3, \tag{20b}$$

where $C_{11}^{\mathrm{M}}, C_{12}^{\mathrm{M}}, \ldots, C_{13}^{\mathrm{F}}$, are the appropriate matrix and fiber strain concentration factors. Using the definitions of Poisson's ratios $\bar{\nu}_{21} = -d\bar{\varepsilon}_2/d\bar{\varepsilon}_1$ and $\bar{\nu}_{31} = -d\bar{\varepsilon}_3/d\bar{\varepsilon}_1$, eqns (20) can be rewritten in the simplified form:

$$d\bar{\varepsilon}_1^{\mathrm{M}} = C_1^{\mathrm{M}}\, d\bar{\varepsilon}_1, \tag{21a}$$

$$d\bar{\varepsilon}_1^{\mathrm{F}} = C_1^{\mathrm{F}}\, d\bar{\varepsilon}_1, \tag{21b}$$

where the modified strain concentration factors C_1^{M} and C_1^{F} are given by

$$C_1^{\mathrm{M}} = C_{11}^{\mathrm{M}} - \bar{\nu}_{21} C_{12}^{\mathrm{M}} - \bar{\nu}_{31} C_{13}^{\mathrm{M}}, \tag{22a}$$

$$C_1^{\mathrm{F}} = C_{11}^{\mathrm{F}} - \bar{\nu}_{21} C_{12}^{\mathrm{F}} - \bar{\nu}_{31} C_{13}^{\mathrm{F}}. \tag{22b}$$

Similarly, one can write the following relations for the transverse strains:

$$d\bar{\varepsilon}_2^{\mathrm{M}} = C_2^{\mathrm{M}}\, d\bar{\varepsilon}_2, \tag{23a}$$

$$d\bar{\varepsilon}_3^{\mathrm{M}} = C_3^{\mathrm{M}}\, d\bar{\varepsilon}_3, \tag{23b}$$

$$d\bar{\varepsilon}_2^{\mathrm{F}} = C_2^{\mathrm{F}}\, d\bar{\varepsilon}_2, \tag{23c}$$

$$d\bar{\varepsilon}_3^{\mathrm{F}} = C_3^{\mathrm{F}}\, d\bar{\varepsilon}_3, \tag{23d}$$

where C_2^{M}, C_3^{M}, C_2^{F} and C_3^{F} are modified strain concentration factors having expressions similar to those of eqns (22).

The strain concentration factors can be obtained from the solution of an appropriate inclusion problem. However, in this work a much simpler approach is followed. This approach is based on deriving a relation between the strain and stress concentration factors as follows. Starting with the expression $d\bar{\sigma}\, d\bar{\varepsilon}_1$ and expanding it in terms of local axial stresses and strains using eqn (2) and a similar equation for the effective overall axial strain, one obtains:

$$d\bar{\sigma}\, d\bar{\varepsilon}_1 = (\bar{c}^{\mathrm{M}}\, d\bar{\sigma}^{\mathrm{M}} + \bar{c}^{\mathrm{F}}\, d\bar{\sigma}^{\mathrm{F}})(\bar{c}^{\mathrm{M}}\, d\bar{\varepsilon}_1^{\mathrm{M}} + \bar{c}^{\mathrm{F}}\, d\bar{\varepsilon}_1^{\mathrm{F}}). \tag{24}$$

Substituting for the effective local stresses and strains from eqns (11) and (21) into eqn (24), and simplifying the result, one obtains the following constraint equation regarding the concentration factors for stresses and strains:

$$(\bar{c}^M B^M + \bar{c}^F B^F)(\bar{c}^M C_1^M + \bar{c}^F C_1^F) = 1. \tag{25a}$$

In view of the constraint relation (12), the above constraint relation can be further simplified to:

$$\bar{c}^M C_1^M + \bar{c}^F C_1^F = 1. \tag{25b}$$

Therefore, once the stress concentration factors B^M and B^F are determined, one can use eqns (25) to derive suitable expressions for the strain concentration factors C_1^M and C_1^F.

In order to formulate the transformation equations for the local axial strain increments $d\varepsilon_1^M$ and $d\varepsilon_1^F$, one uses the hypothesis of elastic energy equivalence using local quantities. Therefore, eqn (17) is rewritten in the form:

$$\tfrac{1}{2}\,d\sigma^M\,d\varepsilon_1^M + \tfrac{1}{2}\,d\sigma^F\,d\varepsilon_1^F = \tfrac{1}{2}\,d\bar{\sigma}^M\,d\bar{\varepsilon}_1^M + \tfrac{1}{2}\,d\bar{\sigma}^F\,d\bar{\varepsilon}_1^F. \tag{26}$$

Substituting for $d\bar{\sigma}^M$ and $d\bar{\sigma}^F$ from eqns (10) into eqn (26), one obtains the following relation between the local axial strain increments and their effective counterparts:

$$d\sigma^M\,d\varepsilon_1^M + d\sigma^F\,d\varepsilon_1^F = \frac{d\sigma^M\,d\bar{\varepsilon}_1^M}{1-\phi_1^M} + \frac{d\sigma^F\,d\bar{\varepsilon}_1^F}{1-\phi_1^F}. \tag{27}$$

After studying eqn (27), it is noticed that it is difficult to derive explicit formulae for $d\bar{\varepsilon}_1^M$ and $d\bar{\varepsilon}_1^F$ without making an assumption. One is led directly to assume local axial strain relations similar to the overall axial strain relation given by eqn (18). Assuming that

$$d\bar{\varepsilon}_1^M = (1-\phi_1^M)\,d\varepsilon_1^M, \tag{28a}$$

$$d\bar{\varepsilon}_1^F = (1-\phi_1^F)\,d\varepsilon_1^F, \tag{28b}$$

one concludes directly that these relations satisfy eqn (27). Similar relations can be assumed for the local–overall transverse strains as those of eqns (19) by replacing ϕ_2 by ϕ_2^M or ϕ_2^F and replacing ϕ_3 by ϕ_3^M or ϕ_3^F. Substituting for $d\bar{\varepsilon}_1^M$ and $d\bar{\varepsilon}_1^F$ from eqns (28) and for $d\bar{\varepsilon}_1$ from eqn (18) into eqns (21), one obtains the following equations for the local axial strain increments in the damaged state:

$$d\varepsilon_1^M = \bar{C}_1^M\,d\varepsilon_1, \tag{29a}$$

$$d\varepsilon_1^F = \bar{C}_1^F\,d\varepsilon_1, \tag{29b}$$

where the strain concentration factors $\bar{C}_1^M$ and $\bar{C}_1^F$ are now defined in the damaged lamina (that is, these are damaged strain concentration factors) and are given by:

$$\bar{C}_1^M = C_1^M\,\frac{1-\phi_1}{1-\phi_1^M}, \tag{30a}$$

$$\bar{C}_1^F = C_1^F\,\frac{1-\phi_1}{1-\phi_1^F}. \tag{30b}$$

Equations (30) can be investigated in a similar way to those of eqns (16) and some figures can be similarly obtained. However, this is not shown here since the resulting figures will be somewhat similar to Figs 3 and 4 and there is no need to repeat them here.

Similarly, using eqns (23) and the appropriate transformation equations for the transverse strains, one obtains:

$$d\varepsilon_2^M = \bar{C}_2^M\,d\varepsilon_2, \tag{31a}$$

$$d\varepsilon_3^M = \bar{C}_3^M\,d\varepsilon_3, \tag{31b}$$

$$d\varepsilon_2^F = \bar{C}_2^F \, d\varepsilon_2, \tag{31c}$$

$$d\varepsilon_3^F = \bar{C}_3^F \, d\varepsilon_3, \tag{31d}$$

where $\bar{C}_2^M$, $\bar{C}_3^M$, $\bar{C}_2^F$ and $\bar{C}_3^F$ are related to C_2^M, C_3^M, C_2^F and C_3^F by the local damage variables. Using generalized forms of eqns (22), one can show that:

$$\bar{C}_{ij}^M = C_{ij}^M \frac{1-\phi_j}{1-\phi_i^M}, \quad i,j = 1,2,3, \tag{32a}$$

$$\bar{C}_{ij}^F = C_{ij}^F \frac{1-\phi_j}{1-\phi_i^M}, \quad i,j = 1,2,3. \tag{32b}$$

Substituting eqns (16) and (30) into eqn (22) and using eqn (15), one obtains the following constraint relation for the damaged stress and strain concentration factors:

$$c^M \bar{C}_1^M + c^F \bar{C}_1^F = 1. \tag{33}$$

Equations (29) provide the required local–overall strain relations that are needed in the next section in order to formulate the damage constitutive equations.

In general, one can show that the constraint relations for the strain concentration factors, appearing partially in eqns (20), take the following form:

$$\bar{c}^M C_{ij}^M + \bar{c}^F C_{ij}^F = \delta_{ij}, \quad i,j = 1,2,3, \tag{34a}$$

$$c^M \bar{C}_{ij}^M + c^F \bar{C}_{ij}^F = \delta_{ij}, \quad i,j = 1,2,3, \tag{34b}$$

where δ_{ij} is equal to 1 when $i = j$ and 0 when $i \neq j$.

2.3. *Constitutive relations*

The elastic constitutive relations are now developed in both the damaged and undamaged states. In addition, the local–overall constitutive relations are also discussed. In the fictitious undamaged lamina, the overall strain increments are given by

$$d\bar{\varepsilon}_1 = \frac{d\bar{\sigma}}{E}, \tag{35a}$$

$$d\bar{\varepsilon}_2 = -\frac{\nu_{21}\, d\bar{\sigma}}{E}, \tag{35b}$$

$$d\bar{\varepsilon}_3 = -\frac{\nu_{31}\, d\bar{\sigma}}{E}, \tag{35c}$$

where the constants E, ν_{21} and ν_{31} are the overall Young's modulus of elasticity and overall Poisson's ratios, respectively. Based on eqns (35), one can write a similar set of overall constitutive relations in the damaged lamina as follows:

$$d\varepsilon_1 = \frac{d\sigma}{\bar{E}}, \tag{36a}$$

$$d\varepsilon_2 = -\frac{\bar{\nu}_{21}\, d\sigma}{\bar{E}}, \tag{36b}$$

$$d\varepsilon_3 = -\frac{\bar{\nu}_{31}\, d\sigma}{\bar{E}}, \tag{36c}$$

where $\bar{E}$, $\bar{\nu}_{21}$ and $\bar{\nu}_{31}$ are the damaged overall Young's modulus of elasticity and Poisson's ratios, respectively. It is noted that $\bar{E}$, $\bar{\nu}_{21}$ and $\bar{\nu}_{31}$ are no longer constants but depend on the damage variables. In order to demonstrate this, one substitutes for $d\bar{\varepsilon}_1$ and $d\bar{\sigma}$ from eqns (18) and (4), respectively, into eqn (35a) and compares the result with eqn (36a). It follows that

$$\bar{E} = E(1-\phi_1)^2. \tag{37}$$

Similarly, substituting for $d\bar{\varepsilon}_2$ and $d\bar{\varepsilon}_3$ from eqns (19) and for $d\bar{\sigma}$ from eqn (4) into eqns (35b) and (35c), and comparing the results with eqns (36b) and (36c), one then obtains:

$$\bar{\nu}_{21} = \nu_{21}\frac{1-\phi_1}{1-\phi_2}, \tag{38a}$$

$$\bar{\nu}_{31} = \nu_{31}\frac{1-\phi_1}{1-\phi_3}. \tag{38b}$$

Alternatively, solving eqns (37) and (38) for the three damage variables ϕ_1, ϕ_2 and ϕ_3, one obtains:

$$\phi_1 = 1-\sqrt{\frac{\bar{E}}{E}}, \tag{39a}$$

$$\phi_2 = 1-\frac{\nu_{21}}{\bar{\nu}_{21}}\sqrt{\frac{\bar{E}}{E}}, \tag{39b}$$

$$\phi_3 = 1-\frac{\nu_{31}}{\bar{\nu}_{31}}\sqrt{\frac{\bar{E}}{E}}. \tag{39c}$$

Equations (39b) and (39c) may be viewed as suitable definitions for the transfer damage variables ϕ_2 and ϕ_3 for this problem. However, generalization of these definitions to other states of deformation and damage is not possible. In general, a fourth-rank damage effect tensor should be considered but this is beyond the scope of this work [for more details, see Murakami (1988), Kattan and Voyiadjis (1990) and Voyiadjis and Kattan (1992)].

It should be mentioned that eqns (37), (38) and (39) are available in the literature (Chow and Wang, 1987). Next, the more difficult task of developing similar relations on the local level as well as the local–overall constitutive relations is considered.

The local elastic stress–strain relations for the fibers and matrix along the fiber direction are given now in the fictitious undamaged configuration:

$$d\bar{\sigma}^{M} = E^{M}\, d\bar{\varepsilon}_1^{M}, \tag{40a}$$

$$d\bar{\sigma}^{F} = E^{F}\, d\bar{\varepsilon}_1^{F}, \tag{40b}$$

where E^M and E^F are the constant moduli of elasticity for the matrix and fiber materials, respectively. Substituting for $d\bar{\sigma}^M$ and $d\bar{\sigma}^F$ from eqns (10) and for $d\bar{\varepsilon}_1^M$ and $d\bar{\varepsilon}_1^F$ from eqns (28) into eqns (40), one obtains:

$$d\sigma^{M} = \bar{E}^{M}\, d\varepsilon_1^{M}, \tag{41a}$$

$$d\sigma^{F} = \bar{E}^{F}\, d\varepsilon_1^{F}, \tag{41b}$$

where $\bar{E}^M$ and $\bar{E}^F$ are the damaged moduli of elasticity given by:

$$\bar{E}^{M} = E^{M}(1-\phi_1^{M})^2, \tag{42a}$$

$$\bar{E}^{F} = E^{F}(1-\phi_1^{F})^2. \tag{42b}$$

Equations (41) represent the local elastic stress–strain relations for the matrix and fibers in the damaged state of the lamina.

Finally the local–overall relations for the moduli of elasticity are now presented. Substituting for $d\sigma^M$ and $d\sigma^F$ from eqns (41), for $d\varepsilon_1^M$ and $d\varepsilon_1^F$ from eqns (29) and for $d\sigma$ from eqn (36a) into eqn (1), one obtains:

$$\bar{E} = c^M \bar{E}^M \bar{C}_1^M + c^F \bar{E}^F \bar{C}_1^F. \tag{43}$$

Performing similar substitutions using eqns (40), (21) and (35a) along with eqn (2), one obtains:

$$E = \bar{c}^M E^M C_1^M + \bar{c}^F E^F C_1^F. \tag{44}$$

Equations (43) and (44) are equivalent when one considers the transformation relations for $\bar{E}$, $\bar{E}^M$, $\bar{E}^F$, $\bar{C}_1^M$, $\bar{C}_1^F$, $\bar{c}^M$ and $\bar{c}^F$ given by eqns (37), (42), (30) and (6). Using eqns (6) and substituting them into eqn (44), one obtains the following expression for the overall elasticity modulus E in terms of the local parameters and the overall damage variable ϕ_1:

$$E = \frac{c^M(1-\phi_1^M)E^M C_1^M + c^F(1-\phi_1^F)E^F C_1^F}{1-\phi_1}. \tag{45}$$

Alternatively, substituting for E from eqn (37) into eqn (45), one obtains the following expression for $\bar{E}$:

$$\bar{E} = c^M E^M C_1^M (1-\phi_1^M)(1-\phi_1) + c^F E^F C_1^F (1-\phi_1^F)(1-\phi_1). \tag{46}$$

The above expression for $\bar{E}$ can also be derived from eqn (43). Equations (45) and (46) represent local–overall relations for the modulus of elasticity.

Using similar relations for the local transverse strains as those of eqns (35b), (35c), (36b) and (36c), one can easily prove the following:

$$\bar{v}_{21}^M = v_{21}^M \frac{1-\phi_1^M}{1-\phi_2^M}, \tag{47a}$$

$$\bar{v}_{21}^F = v_{21}^F \frac{1-\phi_1^F}{1-\phi_2^F}, \tag{47b}$$

where v_{21}^M and v_{21}^F are Poisson's ratios for the matrix and fiber material, respectively. Similar expressions exist for $\bar{v}_{31}^M$ and $\bar{v}_{31}^F$. Similarly, one can derive relations for the local damage variables ϕ_1^M, ϕ_2^M, ϕ_3^M, ϕ_1^F, ϕ_2^F and ϕ_3^F similar to those of eqns (39) with all overall quantities replaced by their local counterparts. Finally, one can derive the following overall–local relations for Poisson's ratios by using eqns (1) and (2) and substituting the transverse strain increments for the stress increments:

$$\frac{c^M E^M C_1^M(1-\phi_1^M) + c^F E^F C_1^F(1-\phi_1^F)}{v_{21}} = \frac{c^M E^M C_2^M(1-\phi_1^M)}{v_{21}^M} + \frac{c^F E^F C_2^F(1-\phi_1^F)}{v_{21}^F} \tag{48a}$$

$$\frac{\bar{E}}{\bar{v}_{21}} = \frac{c^M \bar{E}^M \bar{C}_2^M}{\bar{v}_{21}^M} + \frac{c^F \bar{E}^F \bar{C}_2^F}{\bar{v}_{21}^F}. \tag{48b}$$

Equations (48) are the transverse local–overall relations for Poisson's ratio v_{21} in both the damaged and undamaged configurations. In view of the definition of the matrix Poisson's ratio $\bar{v}_{21}^M = -d\bar{\varepsilon}_2^M/d\bar{\varepsilon}_1^M$ and eqns (21a) and (23a), one can show that $C_1^M \bar{v}_{21}^M = C_2^M \bar{v}_{21}^M$. Similarly, one can show that $\bar{C}_1^M v_{21}^M = \bar{C}_2^M v_{21}$. These two relations can be substituted into eqns (48) appropriately to show that the two equations (48a) and (48b) are equivalent. It should also be noted that similar relations can be shown to exist for Poisson's ratio v_{31}.

The rest of this section is left for a brief discussion of the stress and strain concentration factors B^M, B^F, C_1^M and C_1^F.

In order to determine the concentration factors, one may use the Voigt model (Dvorak and Bahei-El-Din, 1979; Bahei-El-Din and Dvorak, 1989). In this model, it is assumed that the phase strain increments are equal to the overall strain increment. This assumption will be applied here to the undamaged state, that is $d\bar{\varepsilon}_1^M = d\bar{\varepsilon}_1^F = d\bar{\varepsilon}_1$. Incorporating this assumption into the presented theory by comparing with eqns (21), one directly concludes that $C_1^M = C_1^F = 1$. Upon further using eqns (40), one has $d\bar{\sigma}^M = E^M\, d\bar{\varepsilon}_1 = E^M\, d\bar{\sigma}/E$. Comparing this with equation (11a), one concludes that $B^M = E^M/E$. A similar argument shows that $B^F = E^F/E$.

The reader should be cautious, however, in using this assumption. Although the expressions obtained for the stress and strain concentration factors are very simple, there are certain inconsistencies that arise as a result of adopting this assumption. For example, using a local relation for the matrix similar to that of eqn (35b), one has $d\sigma^M = (-E^M/\nu_{21}^M)\, d\bar{\varepsilon}_2^M = (E^M \nu_{21}/E\nu_{21}^M)\, d\bar{\sigma}$. Comparing this with eqn (11a) and the above result for B^M, one concludes that $\nu_{21}^M = \nu_{21}$. This is obviously a contradiction since the matrix and overall Poisson's ratios are different. This contradiction arises directly from the simple assumption of the Voigt model. In addition, the derived expressions for the concentration factors using this model violate the constraint equations (12) and (25b). Other more realistic models for determining the concentration factors are available, however they are far from being simple.

The above contradiction can be corrected by employing the Vanishing Fiber Diameter (VFD) model (Dvorak and Bahei-El-Din, 1979; Bahei-El-Din and Dvorak, 1989). In this model, it is assumed that each of the cylindrical fibers has a vanishing diameter and that the fibers occupy a finite volume fraction of the composite [in order to provide axial constraint of the phase, Dvorak and Bahei-El-Din (1979, 1982)]. For the problem considered here, these assumptions reduce to

$$d\bar{\sigma} = c^M\, d\bar{\sigma}^M + c^F\, d\bar{\sigma}^F, \tag{49a}$$

$$d\bar{\varepsilon}_1 = d\bar{\varepsilon}_1^M = d\bar{\varepsilon}_1^F, \tag{49b}$$

$$d\bar{\varepsilon}_2 = c^M\, d\bar{\varepsilon}_2^M + c^F\, d\bar{\varepsilon}_2^F \tag{49c}$$

and

$$d\bar{\varepsilon}_3 = c^M\, d\bar{\varepsilon}_3^M + c^F\, d\varepsilon_3^F. \tag{49d}$$

It is clear that the axial strain increment assumption (49b) conforms with that of the Voigt model. However, a more realistic assumption is provided for the transverse strain increments (49c) and (49d) which is compatible with the physics of the problem. Considering the argument of the previous paragraph, it can be seen that the contradiction concerning Poisson's ratio no longer exists in the VFD model and therefore this model is appropriate to use for this problem.

2.4. *Damage evolution*

There are several approaches in the literature on the topic of evolution of damage and the proper form of the kinetic equation of the damage variable. Kachanov (1986) proposed an evolution of damage based on a power law with two independent material constants. However, adopting such a law here for each of the matrix and fiber materials would leave four independent material constants to be determined. In addition, the resulting overall kinetic equation for damage evolution cannot be solved. Therefore, a more rational approach based on energy considerations will be adopted here.

The approach followed here will depend on the introduction of a damage strengthening criterion in terms of a function g, and a generalized thermodynamic force that corresponds to the damage variable ϕ_1 (Lemaitre, 1985; Lee *et al.*, 1985). The elastic strain energy U in the damaged composite system is given by

$$U = \tfrac{1}{2}\bar{E}\varepsilon_1^2 = \tfrac{1}{2}E(1-\phi_1)^2\varepsilon_1^2. \tag{50}$$

Therefore, the incremental strain energy $\mathrm{d}U$ is given by

$$\mathrm{d}U = E(1-\phi_1)^2\varepsilon_1\,\mathrm{d}\varepsilon_1 - E(1-\phi_1)\varepsilon_1^2\,\mathrm{d}\phi_1. \tag{51}$$

The generalized thermodynamic force y_1 associated with the overall damage variable ϕ_1 is thus defined by

$$y_1 = \frac{\partial U}{\partial \phi_1} = -E(1-\phi_1)\varepsilon_1^2. \tag{52}$$

Let $g(y_1, L)$ be the damage function (criterion) as proposed by Lee *et al.* (1985), where $L = L(\beta)$ is a damage strengthening parameter which is a function of the overall damage parameter β. For this problem, the function g takes the following form:

$$g = \tfrac{1}{2}y_1^2 - L(\beta) \equiv 0. \tag{53}$$

In order to derive a normality rule for the evolution of damage, one first starts with the power of dissipation Π which is given by

$$\Pi = -y_1\dot{\phi}_1 - L\dot{\beta}, \tag{54}$$

where a superposed dot indicates material time derivative. The problem is to extremize Π subject to the condition $g = 0$. Using the theory of functions of several variables, one introduces the Lagrange multiplier $\dot{\lambda}$ and forms the function $H(y_1, L)$ such that

$$H = \Pi - \dot{\lambda}g. \tag{55}$$

The problem now reduces to extremizing the function H. For this purpose, the two necessary conditions are $\partial H/\partial y_1 = 0$ and $\partial H/\partial L = 0$. Using these conditions along with eqns (54) and (55), one obtains

$$\dot{\phi}_1 = -\dot{\lambda}\frac{\partial g}{\partial y_1} \tag{56a}$$

$$\dot{\beta} = -\dot{\lambda}\frac{\partial g}{\partial L}. \tag{56b}$$

Substituting for g from eqn (53) into eqn (56b), one concludes directly that $\dot{\lambda} = \dot{\beta}$. Substituting this into eqn (56a), along with eqn (53), one obtains:

$$\dot{\phi}_1 = -\dot{\lambda}y_1. \tag{57}$$

In order to solve the differential equation (57), one must first find an expression for the Lagrange multiplier $\dot{\lambda}$. This can be obtained by invoking the consistency condition $\dot{g} = 0$. Therefore, one obtains:

$$\frac{\partial g}{\partial y_1}\dot{y}_1 + \frac{\partial g}{\partial L}\dot{L} = 0. \tag{58}$$

Substituting for $\partial g/\partial y_1$ and $\partial g/\partial L$ from eqn (53) and for $\dot{L} = \dot{\beta}\,\partial L/\partial \beta$ (from the chain rule), and solving for $\dot{\beta}$, one obtains:

$$\dot{\beta} = \dot{\lambda} = \frac{y_1 \dot{y}_1}{\partial L/\partial \beta}, \tag{59}$$

substituting the above expression of $\dot{\lambda}$ into (57), one obtains the kinetic (evolution) equation of overall damage:

$$\left(\frac{\partial L}{\partial \beta}\right)\dot{\phi}_1 = -y_1^2 \dot{y}_1, \tag{60}$$

with the initial condition that $\phi_1 = 0$ when $y_1 = 0$. The solution of eqn (60) depends on the form of the function $L(\beta)$. For simplicity, one may consider a linear function in the form $L(\beta) = c\beta + d$, where c and d are constants. This is motivated by the hardening parameter defined for isotropic hardening in plasticity as $\sqrt{\dot{\varepsilon}''_{ij}\dot{\varepsilon}''_{ij}}$ where $\dot{\varepsilon}''_{ij}$ is the plastic component of the strain rate. The equivalent damage strengthening parameter can be analogously expressed as $\sqrt{\dot{\beta}\dot{\beta}}$ or simply $\dot{\beta}$ whereby giving a linear function in β as discussed above. Substituting this into eqn (60) and integrating, one obtains the following relation between the overall damage variable ϕ_1 and its associated generalized force y_1:

$$\phi_1 = -\frac{y_1^3}{3c}. \tag{61}$$

The above relation is shown in Fig. 5 where it is clear that ϕ_1 is a monotonically increasing function of y_1.

Next, one investigates the overall strain–damage relationship. Differentiating the expression of y_1 in eqn (52), one obtains:

$$\dot{y}_1 = E\varepsilon_1[\varepsilon_1\dot{\phi}_1 - 2\dot{\varepsilon}_1(1-\phi_1)]. \tag{62}$$

Substituting the expressions of y_1 and $\dot{y}_1$ of eqns (52) and (62), respectively, into eqn (60), one obtains the strain–damage differential equation:

$$\left(\frac{\partial L}{\partial \beta}\right)\dot{\phi}_1 = E^3\varepsilon_1^5(1-\phi_1)^2[2\dot{\varepsilon}_1(1-\phi_1) - \varepsilon_1\dot{\phi}_1]. \tag{63}$$

The above differential equation can be solved easily by the simple change of variables $x = \varepsilon_1^2(1-\phi_1)$ and noting that the expression on the right-hand side is nothing but $E^3x^2\dot{x}$.

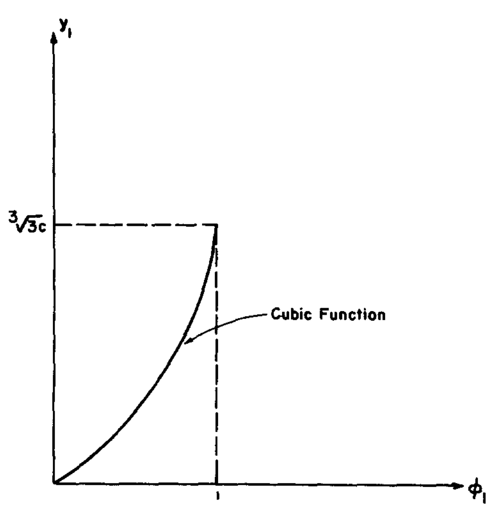

Fig. 5. Relation between the overall damage variables ϕ_1 and its associated generalized force y_1.

Performing the integration with the initial condition that $\phi_1 = 0$ when $\varepsilon_1 = 0$ along with the linear expression of $L(\beta)$, one obtains:

$$\frac{\phi_1}{(1-\phi_1)^3} = \frac{E^3}{3c}\varepsilon_1^6. \tag{64}$$

One should note that an initial condition involving an initial damage variable ϕ_1^0 could have been used, i.e. $\phi_1 = \phi_1^0$ when $\varepsilon_1 = 0$. The strain–damage relation of eqn (64) could easily have been obtained by substituting the expression of y_1 of eqn (52) directly into eqn (61). However, it is preferable to derive it directly from the strain–damage differential equation (63) without the use of the generalized force y_1.

One can now easily incorporate local damage evolution for the composite based on the previous discussion. One assumes that there exist two local damage strengthening criteria $g^M(y_1^M, L^M)$ and $g^F(y_1^F, L^F)$ having the same forms as that of eqn (53), where y_1^M and y_1^F are the generalized thermodynamic forces associated with ϕ_1^M and ϕ_1^F, respectively, and L^M and L^F are the local counterparts of L. Linear expressions are also assumed for L^M and L^F such that $L^M = c_1\beta^M + d_1$ and $L^F = c_2\beta^F + d_2$, where β^M and β^F are local counterparts of β and c_1, c_2, d_1, d_2 are constants.

Assuming matrix and fiber damage evolution laws similar to that of eqns (60) and (61), one can write

$$\phi_1^M = \frac{(y_1^M)^3}{3c_1}, \tag{65a}$$

$$\phi_1^F = \frac{(y_1^F)^3}{3c_2}. \tag{65b}$$

Substituting eqns (61) and (65) into eqn (8) and simplifying the result, one obtains the local–overall relation for the generalized thermodynamic force associated with the damage variable:

$$y_1^3 = c\left[\frac{c^M}{c_1}(y_1^M)^3 + \frac{c^F}{c_2}(y_1^F)^3\right]. \tag{66}$$

Finally, using the above equation along with the fact that $y_1 = \partial g/\partial y_1$ and similar expressions for y_1^M and y_1^F, one obtains:

$$\left(\frac{\partial g}{\partial y_1}\right)^3 = c\left[\frac{c^M}{c_1}\left(\frac{\partial g^M}{\partial y_1^M}\right)^3 + \frac{c^F}{c_2}\left(\frac{\partial g^F}{\partial y_1^F}\right)^3\right]. \tag{67}$$

Equation (67) is a nonlinear partial differential equation that represents the local–overall relation for the damage strengthening criteria for the matrix, fibers and the overall composite system.

3. CONCLUSION

A micromechanical damage analysis is proposed for a unidirectional fiber-reinforced composite thin lamina subjected to uniaxial tension. The analysis is based on a combination of the micromechanical composite model coupled with continuum damage mechanics. An overall damage variable is defined for the composite system based on the concept of effective stress. In addition, two local damage variables are introduced to account for the damage induced in the matrix and fibers.

New expressions are derived for the stress and strain concentration factors for the damaged material in terms of the undamaged concentration factors and the damage variables. Finally, a criterion for damage evolution is proposed for the composite system using an extremum principle. The theory presented here can be generalized for general states of deformation and damage in composite materials, however, tensor analysis is needed for the mathematical formulation. Therefore, the generalization of this theory is left to a forthcoming paper.

Acknowledgements—The research described in this paper was sponsored by the Air Force Office of Scientific Research under Grant AFOSR-90-0227. The authors wish to acknowledge the support and encouragement of Dr George K. Haritos and Dr Walter F. Jones.

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